On the informational completeness of local observables

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Motivation

Curse of dimensionality: For problems that involve many degrees of freedom, the dimension of the phase space blows up exponentially.

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  - Performing quantum state verification
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Goal : find a large class of states $\mathcal{S}$ such that

- Some of these tasks can be done efficiently.
- If a state is in $\mathcal{S}$, one can efficiently verify that fact.
- The above features remain robust against imperfect measurements/finite precision.

Brief summary

I will propose a class of states over n particles, $\mathcal{S}_n$ that has the following features.

- One can verify that the state is in $\mathcal{S}_n$ with $O(n)$ measurement/computation time.
- Any state in $\mathcal{S}_n$ is defined by a set of $O(1)$-particle density matrices.
  - State tomography/verification can be done with $O(n)$ measurement/computation time.
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  - State tomography/verification can be done with $O(n)$ measurement/computation time.
  - Small errors in the $O(1)$-particle density matrices don't propagate too much.(robust error bound)
- The class includes highly entangled states(e.g., topological code, quantum Hall system).

Informational completeness

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1. $\mathrm{Tr}(\rho) = 1$.
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Specifying ρ : Assign expectation values for all linearly independent observables.($\approx 4^n$)
Such observables are informationally complete: their expectation values completely determine the state.

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What if we do not specify all the expectation values of the linearly independent observables? : The problem is inherently ill-defined. **Or, is it?**

Informational completeness of local observables

Sometimes, expectation values of local observables completely determine the global state.

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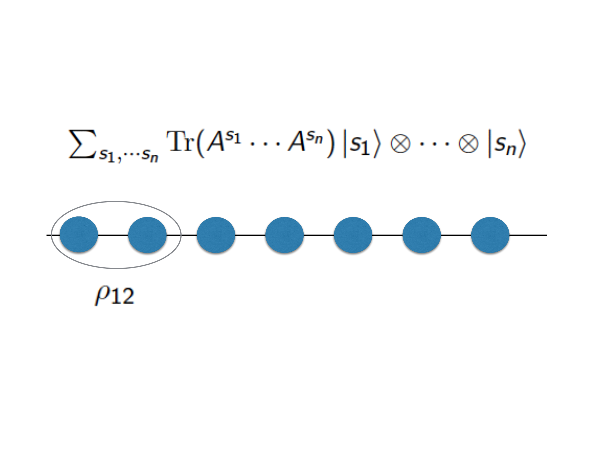
1. Product state : $|\psi\rangle = |0\rangle \otimes |1\rangle \otimes \cdots \otimes |1\rangle$.
2. Matrix product states : $\sum_{s_1,\cdots s_n} \mathrm{Tr}(A^{s_1} \cdots A^{s_n}) |s_1\rangle \otimes \cdots \otimes |s_n\rangle$ [Cramer et al. 2011]

Matrix product states

For (injective) matrix product states, local observables can be informationally complete.



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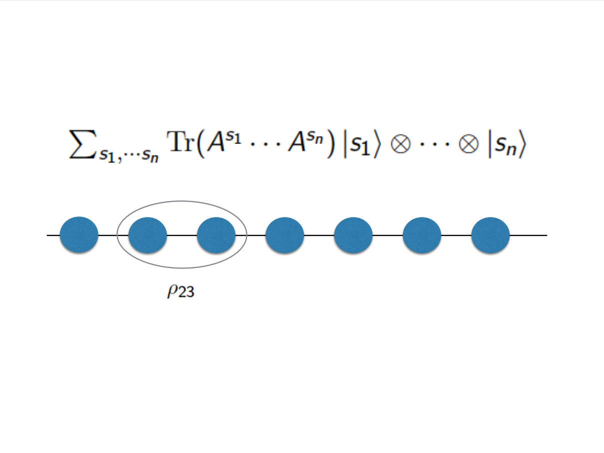

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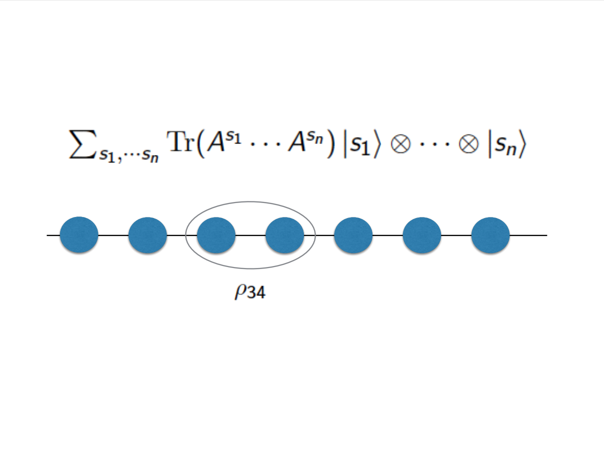

Matrix product states

For (injective) matrix product states, local observables can be informationally complete.

$$\rho_{12}, \rho_{23}, \cdots \rightarrow \text{ MPS tomography algorithm} \rightarrow \text{Output}$$

Output : MPS $|\psi'\rangle$ that is consistent with $\rho_{12}, \rho_{23}, \cdots$ with a **certificate showing that** $|\langle\psi'|\psi_{real}\rangle| \geq 1 - \epsilon$. [Cramer et al. 2011]

Takeaway message

Given a set of expectation values of local observables, there exists an efficiently checkable condition that tells you whether they are informationally complete.

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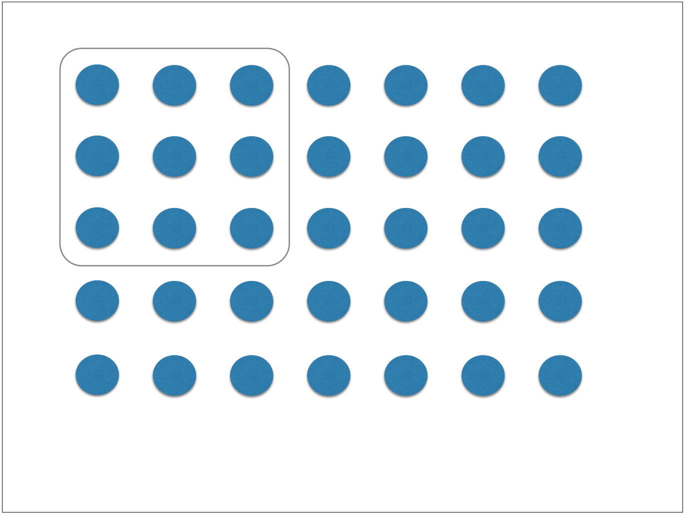
Our result can be thought as a generalization of the result of Cramer *et al.* to higher dimensional systems, but with an important difference.

- Cramer et al. appeals to the special structure of the MPS, but **our approach does not involve any global wavefunction at all.**

Setup



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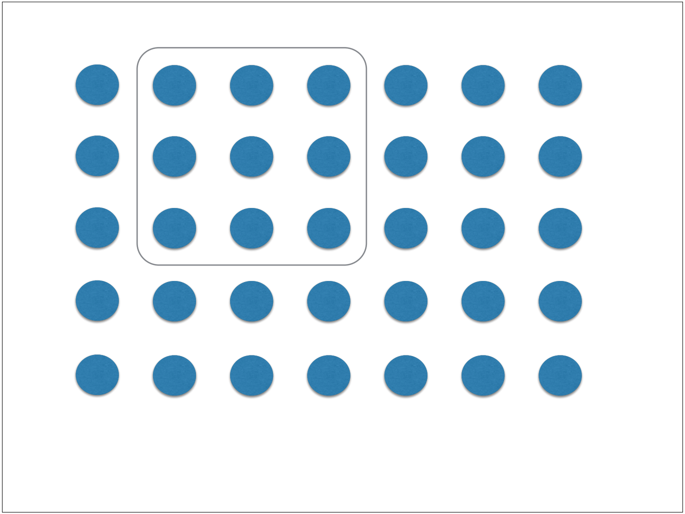

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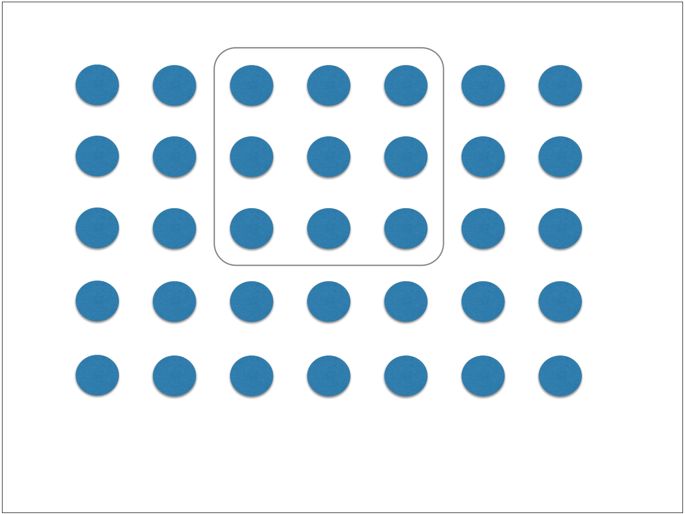

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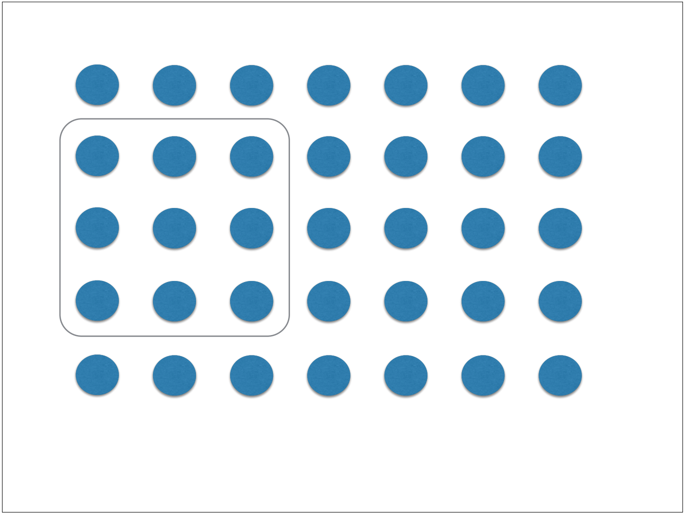

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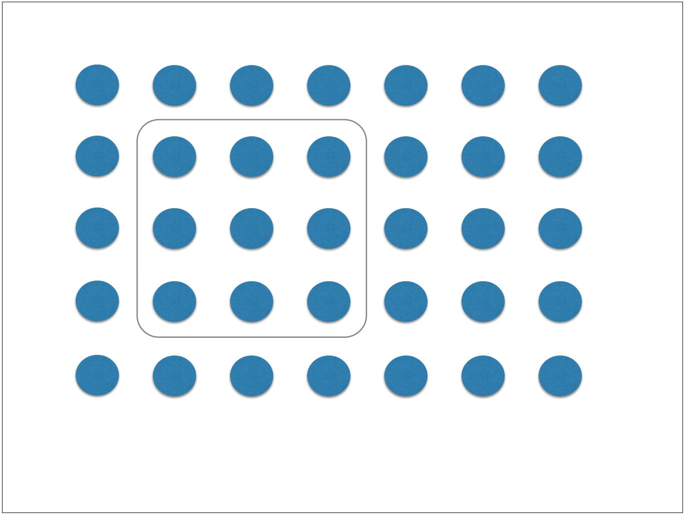

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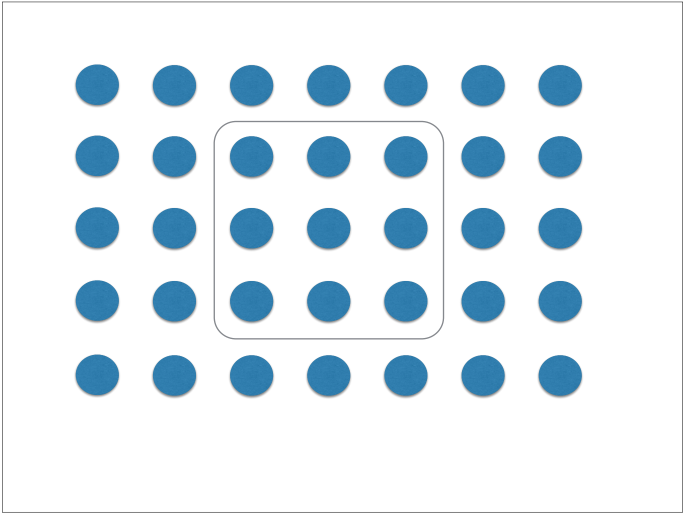
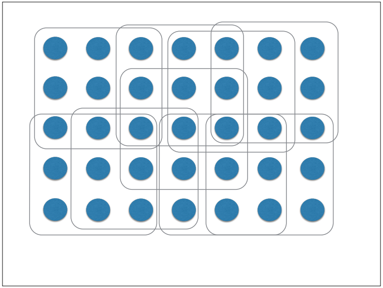


Question



For all sites k, we know the reduced density matrices over the neighborhood of k.

- k : Site
- $\mathcal{N}_k$: Neighborhood of k.

Question: If one can find a state ρ' that is consistent with $\{\rho_{k\mathcal{N}_k}\}$, is it close to ρ?

Main result(Colloquial version)

There exists a **certificate** $\epsilon(\{\rho_{k\mathcal{N}_k}\})$ such that,

$$|\rho - \rho'|_1 \leq \epsilon(\{\rho_{k\mathcal{N}_k}\}).$$

- Efficiency : $O(n)$ measurement/computation time.
- Applicability : any 1D/2D gapped system assuming a certain form of area law holds, but possibly more.
  - Both with and without topological order!
- Robustness: if $|\rho_{k\mathcal{N}_k} - \rho'_{k\mathcal{N}_k}| = \epsilon$, there is an additional error term which is $O(n\epsilon \log \frac{1}{\epsilon})$.

Applications

- Quantum state tomography
- Quantum state verification
- Possibly more?

The plan

Recall our main result:

$$|\rho - \rho'|_1 \leq \epsilon(\{\rho_{k\mathcal{N}_k}\})$$

1. Globally Computable Upper Bound : I will upper bound a trace distance between ρ and ρ' by a quantity that can be computed from the global states.
2. Locally Computable Upper Bound : Using information inequalities, I will introduce a new quantity that can be computed from local reduced density matrices. This quantity will be an upper bound of the GCUB.
3. I will show that the LCUB is small for many systems.

Globally Computable Upper Bound : background

$I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)$ is quantum conditional mutual information.

$$I(A:C|B) \geq 0.$$

[Lieb, Ruskai 1972]
*$S(A) = -\mathrm{Tr}(\rho_A \log \rho_A)$: entanglement entropy of A.

Globally Computable Upper Bound : background

[Petz, 1984]

$$I(A:C|B) = 0$$

if and only if

$$\rho_{ABC} = \rho_{BC}^{\frac{1}{2}} \rho_B^{-\frac{1}{2}} \rho_{AB} \rho_B^{-\frac{1}{2}} \rho_{BC}^{\frac{1}{2}}.$$

* The precise form of the equation does not matter for the purpose of this talk. **What matters is the fact that the global state is completely determined by its local reduced density matrices, if the conditional mutual information is 0.**

Globally Computable Upper Bound : background

Suppose ρ_{ABC} and σ_{ABC} are locally consistent, *i.e.*,

$$\rho_{AB} = \sigma_{AB}, \quad \rho_{BC} = \sigma_{BC},$$

and $I(A:C|B)_\rho = I(A:C|B)_\sigma = 0$.

$$\rho_{ABC} = \sigma_{ABC} = \rho_{BC}^{\frac{1}{2}} \rho_B^{-\frac{1}{2}} \rho_{AB} \rho_B^{-\frac{1}{2}} \rho_{BC}^{\frac{1}{2}}.$$

If ρ_{ABC} and σ_{ABC} are conditionally independent, and their marginal distributions over AB and BC are consistent, they are globally equivalent.

Globally Computable Upper Bound

Colloquially : If ρ_{ABC} and σ_{ABC} are approximately conditionally independent, *i.e.*,

$$I(A:C|B)_\rho \approx 0, \quad I(A:C|B)_\sigma \approx 0$$

and $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\rho_{ABC} \approx \sigma_{ABC}.$$

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Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

* If $\rho_{AB} \approx \sigma_{AB}$ and $\rho_{BC} \approx \sigma_{BC}$, there is an additional additive contribution proportional to log(*dimension*).

Globally Computable Upper Bound

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

: ρ_{ABC} and σ_{ABC} are close to each other if

- Their marginal distribution over AB and BC are the same, and
- $I(A:C|B)_\rho \approx 0$ and $I(A:C|B)_\sigma$.

Locally Computable Upper Bound

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

Okay that is kind of cool. I guess you are trying to use this result to bound the trace distance between two states from its local reduced density matrices?

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But that is never going to work. You see, in order to compute the upper bound, you need to know the entropy of the global states.

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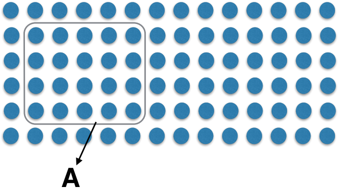
I mean, let's suppose, WLOG, A is a very large region like this.



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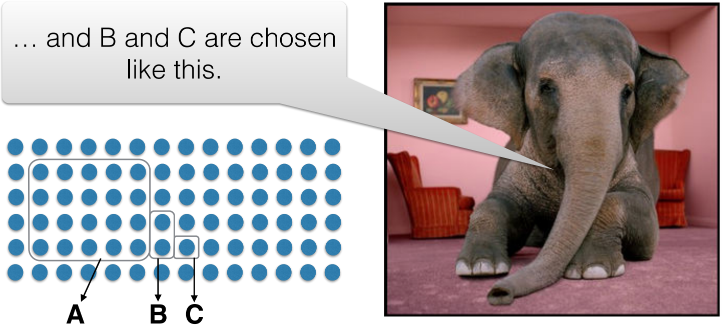

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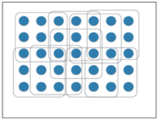
Remember the setup?

Higher dimensions : Setup



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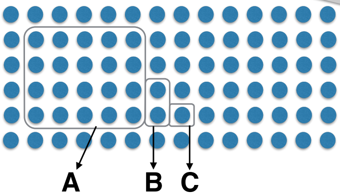
If one can find a state ρ' that is consistent with $\{\rho_{k\mathcal{N}_k}\}$, is it close to ρ?

Locally Computable Upper Bound

T ... ,

$$\ldots |B)_\rho + I(A:C|B)_\sigma).$$

How do you propose to compute I(A:C|B), knowing only the density matrices over each sites and its neighbours?

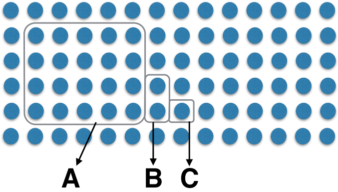


Locally Computable Upper Bound

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

Recall I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)



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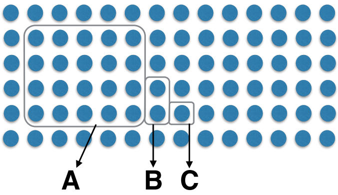
$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

S(BC) and S(B) can be computed easily from the given local density matrices.



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Locally Computable Upper Bound

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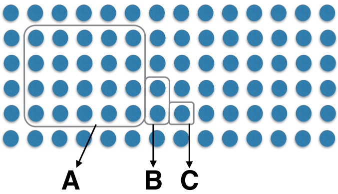
$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

The nontrivial part is the remaining term, S(AB)-S(ABC).



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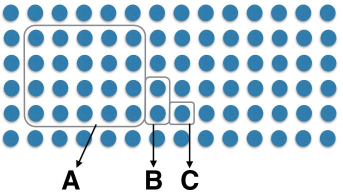
Locally Computable Upper Bound

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$)_\rho + I(A:C|B)_\sigma).$$

I will give you an upper bound on S(AB) - S(ABC) which can be computed from the given reduced density matrices.



I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC)

Locally Computable Upper Bound : Weak monotonicity

Strong subadditivity asserts that

$$S(AB) + S(BC) - S(B) - S(ABC) \geq 0$$

for any tripartite state ρ_{ABC}.
Weak monotonicity asserts that

$$S(DE) - S(D) + S(EF) - S(F) \geq 0$$

for any tripartite state ρ_{DEF}.

Locally Computable Upper Bound : Weak monotonicity

Weak monotonicity asserts that

$$S(DE) - S(D) + S(EF) - S(F) \geq 0$$

for any tripartite state ρ_{DEF}.

$$S(EF) - S(F) \geq S(D) - S(DE).$$

Setting $D = AB, E = C$,

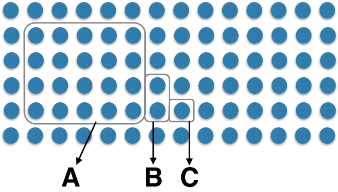
$$S(CF) - S(F) \geq S(AB) - S(ABC).$$

LCUB $\geq$ GCUB

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

For any F, S(CF) - S(F) is larger or equal to S(AB)-S(ABC).



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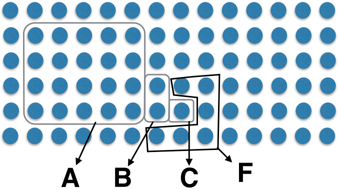
$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

In particular, I can choose F as follows.



$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B)$$

LCUB $\geq$ GCUB

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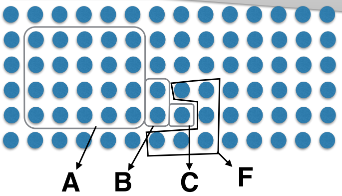
$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

But still, how do you know that S(CF) - S(F) +S(BC)-S(B) is small?



$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B)$$

LCUB $\geq$ GCUB

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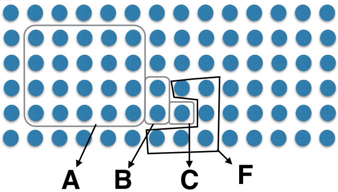
$)_{\sigma})$.

You don't. However, given a set of local reduced density matrices, we can easily check this condition. Further, there is a good reason to believe that the upper bound is close to 0 for gapped systems in 1D and 2D.



$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B)$$

The good reason : **strong** area law

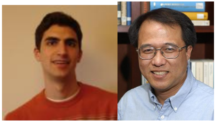
There is a general belief that if a quantum many-body system has a constant energy gap between its ground state sectors and its first excited state, entanglement entropy satisfies area law:

$$S(A) = a|\partial A|^{D-1} + b|\partial A|^{D-2} + \cdots .$$

In particular, in 2D,

$$S(A) = a|\partial A| - \gamma + o(1)$$

(Kitaev and Preskill, Levin and Wen 2006)

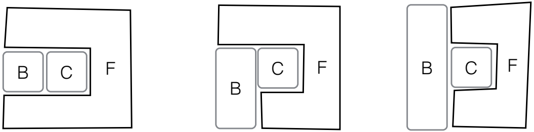


* The above assertion is a much stronger statement than this:

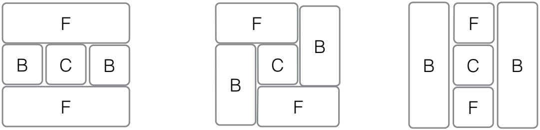
$$S(A) = O(|\partial A|).$$

The upper bound depends on the topology.

Plugging in the entanglement entropy formula,



$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$



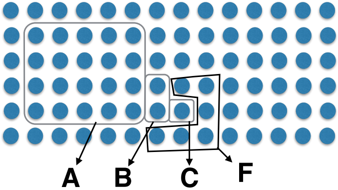
$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B) = 2\gamma + o(1)$$

LCUB $\geq$ GCUB

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq \frac{1}{2}(I(A:C|B)_\rho + I(A:C|B)_\sigma).$$

Plugging in the entanglement entropy formula, we get the desired upper bound.



$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$

Bootstrapping the argument

Suppose two states are consistent over each sites and their neighbours.

Theorem 1. If $\rho_{AB} = \sigma_{AB}$ and $\rho_{BC} = \sigma_{BC}$,

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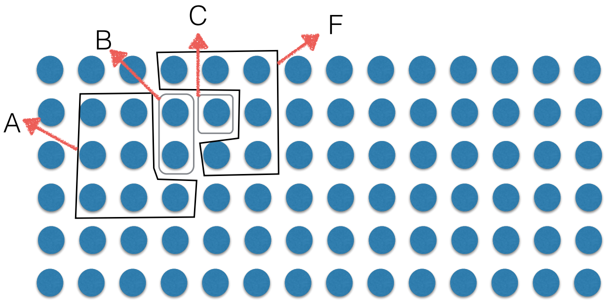

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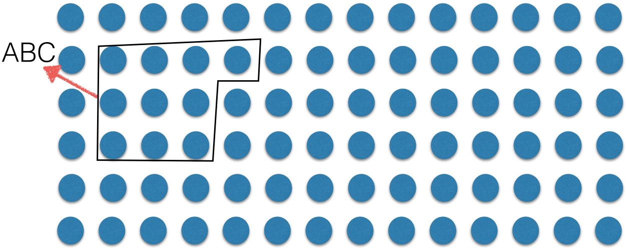

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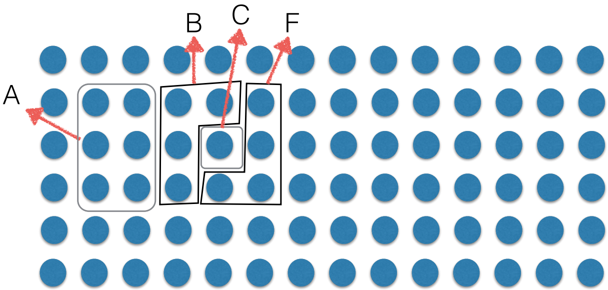

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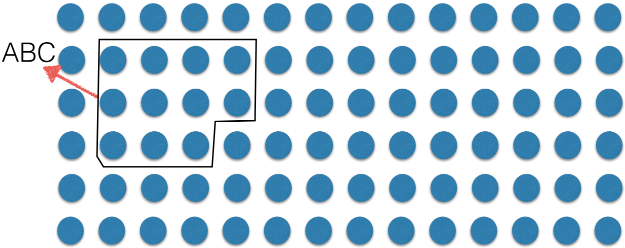

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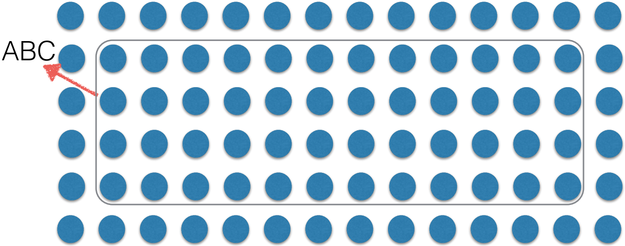

$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B) = o(1)$$

Three key ideas

1. If $\rho_{AB} \approx \sigma_{AB}$, $\rho_{BC} \approx \sigma_{BC}$, $I(A:C|B)_\rho \approx 0$, and $I(A:C|B)_\sigma \approx 0$, then $\rho_{ABC} \approx \sigma_{ABC}$.
2. Independent of the size of A, there is an upper bound on $I(A:C|B)$ that can be computed from the local reduced density matrices.
3. The upper bound is likely to be small for many interesting systems, *e.g.*, gapped systems in 1D/2D.

Application : Quantum state tomography/verification

1. Quantum state tomography : Estimate the local reduced density matrices, find a state consistent with the local reduced density matrices, and then check the locally computable upper bound. If it is close to 0, we are done!
   - Disclaimer: Finding such a state may take a LONG time.
2. Quantum state verification : Estimate the local reduced density matrices, and check the consistency with the target quantum state. If the locally computable upper bound is close to 0, we are done!

Summary

- For a large class of interesting multipartite states, there exists a locally checkable condition, under which the expectation values of certain nonlocal observables are completely determined by the expectation values of the local observables.
- The condition is likely to be satisfied for generic gapped 1D/2D systems.
- For such systems, the number of measurement data that is information-theoretically sufficient to estimate the state grows moderately with the system size.

Comments & Future direction

- The technical part of this work is based on the strong subadditivity of entropy and the concavity of von Neumann entropy.
  - Better bound using generalized entropies(as opposed to the von Neumann entropy)?
- Are there other implications of $I(A:C|B) \approx 0$?
  - See 1410.0664(Fawzi and Renner), 1411.4921(Brandão et al.), 1412.4067(Berta et al.), and references therein.
- The bound itself is applicable to any quantum states(assuming quantum mechanics is right), and it becomes nontrivial under the strong area law assumption.
  - Are there other interesting scenarios under which the bound becomes nontrivial?
- For tomographic application, our result does not provide a method to explicitly write down the global state.
  - But do we really need to write down the global state when we know that the local data determines the global state?

Globally Computable Upper Bound : Proof Idea

Starting from a useful lemma:
Lemma 1. (Kim 2013)

$$\frac{1}{8}|\rho - \sigma|_1^2 \leq S(\frac{\rho + \sigma}{2}) - \frac{S(\rho) + S(\sigma)}{2},$$

we can show

$$\frac{1}{8}|\rho_{ABC} - \sigma_{ABC}|_1^2 \leq S(\frac{\rho_{ABC} + \sigma_{ABC}}{2}) - \frac{S(\rho_{ABC}) + S(\sigma_{ABC})}{2}.$$

By SSA,

$$S(\frac{\rho_{ABC} + \sigma_{ABC}}{2}) \leq S(AB)_{\frac{\rho+\sigma}{2}} + S(BC)_{\frac{\rho+\sigma}{2}} - S(B)_{\frac{\rho+\sigma}{2}}.$$

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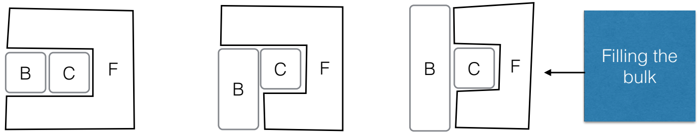
$S(AB)_{\frac{\rho+\sigma}{2}} = S(AB)_\rho = S(AB)_\sigma$, and a similar story for BC, B.

The upper bound depends on the operation.

Plugging in the entanglement entropy formula,



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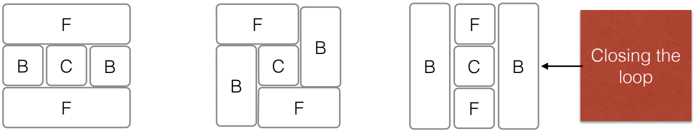

$$I(A:C|B) \leq S(CF) - S(F) + S(BC) - S(B) = \boxed{2\gamma} + o(1)$$

$$\gamma = \sqrt{\sum_a d_a^2}$$ d_a : quantum dimension of a topological charge a.

Local consistency vs. global consistency depends on γ and global topology

- If $\gamma = 0$, an overlapping set of local reduced density matrices completely determine the global state for any compact manifold.
- If $\gamma \neq 0$, an overlapping set of local reduced density matrices completely determine the reduced density matrix over any region that does not contain any logical operator.
  - In particular, an overlapping set of local reduced density matrices completely determine the global state on a sphere.